**Discrete Homotopy on Graphs and Clique Graphs**

Importing the concept from topology, we can define two (reflexive) morphisms of graphs $f, g : X \to Y$ to be *homotopic* ($f \simeq g$) if there is a graph morphism $H : X \boxtimes P_n \to Y$ with $H(x, 1) = f(x)$ and $H(x, n) = g(x)$ (here $P_n$ is the path graph on $n$ vertices). This discrete homotopy of graph morphisms share many formal properties with the homotopy of continuous maps on topological spaces. In particular, $\simeq$ is a congruence relation on the category of graphs $\mathcal{G}$, so we can construct the quotient category $\mathcal{G}/\simeq$. These ideas have been studied before by Babson, Dochterman, Kozlov and Lovàsz.

The clique operator $K$, transforms a graph $X$ into the intersection graph of all its (maximal) cliques $K(X)$. Many papers have been published regarding clique behavior, i.e.: given a graph $X$, determine whether it $K$-converges ($K^n(X) \cong K^m(X)$ for some $n < m$) or $K$-diverges ($\lim_{n\to\infty} |K^n(X)| = \infty$). Here we note that whereas $K$ is not a functor on $\mathcal{G}$, it is indeed a functor on $\mathcal{G}/\simeq$. This fact, gives new insight into the problem of clique behavior and a new vast panorama emerges with new theorems, new open problems, new divergence techniques, and a unifying approach to several existing divergence techniques.

Let us see a concrete example. Given a morphism $f : X \to Y$, we define its norm as $||f|| = \min_{f' \simeq f} |\mathrm{Im}(f')|$ and we say that $f$ is *unbounded* if the set $\{||K^n(f)|| \mid n \in \mathbb{N}\}$ is unbounded. It follows that whenever $f$ is unbounded, both $X$ and $Y$ are $K$-divergent, and whenever $f$ *factorizes* in $\mathcal{G}/\simeq$, i.e. $f \simeq hg$ for some $g : X \to Z$ and $h : Z \to Y$, the morphisms $g$ and $h$ are also unbounded and hence $Z$ is also $K$-divergent. When $X$ and $Y$ are $K$-divergent in $\mathcal{G}/\simeq$, the *retractions* and the *triangular covering maps* are previously studied divergence techniques that become particular cases of unbounded morphisms.